Chapter 5

Series Solutions

5.1 Solutions about Ordinary Points

5.1.1 Review of Power Series

Let us review some properties of **power series**

Definition 5.1.1. Given a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, we say

(1) it **converges** if the series converges at x. Otherwise it **diverges**.

(2) $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ **converges absolutely** at x if $\sum_{n=0}^{\infty} |a_n(x-x_0)^n|$ converges.

Radius of convergence

(1) Given a power series, there exists a nonnegative number ρ such that at least one of the following holds: .

  (a) The power series converges absolutely for $|x-x_0| < \rho$.

  (b) The power series diverges for $|x-x_0| > \rho$.

  (c) When $|x-x_0| = \rho$, the power series may converge or diverge.

  The number $\rho > 0$ is called the **radius of convergence**.

(2) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then $\frac{1}{\rho} = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

(3) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ exists then we have $\frac{1}{\rho} = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$.

(4) The limit $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$ always exists and it holds that

$$\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Here $\limsup_{n\to\infty} b_n := \lim_{n\to\infty} \sup_{k\geq n} b_k$.

(5) If two power series $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ converge on the interval $|x-x_0| < \rho$ then

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$$

holds and the sum also converges for $|x-x_0| < \rho$.

(6) The power series of $f(x) \cdot g(x)$ also converges for $|x-x_0| < \rho$ and if we let $c_n = a_0 b_n + \cdots + a_n b_0$ then we have

$$f(x) \cdot g(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n.$$

(7) If $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for $|x-x_0| < \rho$ we have

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \\ f''(x) &= \sum_{n=2}^{\infty} n(n+1) a_n (x-x_0)^{n-2} \\ &\cdots \\ f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x-x_0)^{n-k} \end{aligned}$$

holds for $|x-x_0| < \rho$.

(8) Set $x = x_0$ we obtain

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

and hence we can write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

This is called the **Taylor's series** of f at $x = x_0$.

(9) (Uniqueness of Taylor's series) If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ on some interval $|x-x_0| < \delta$, then $a_n = b_n$ for all n.

If f has a convergent power series at x_0, then we say f is **analytic** at x_0.

Shift of Index

Example 5.1.2. Shift of index.

$$(1)\ \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n$$

$$(2)\ \sum_{n=2}^{\infty} (n+2)(n+1)a_n(x-x_0)^{n-2} = \sum_{n=0}^{\infty} (n+4)(n+3)a_{n+2}(x-x_0)^n$$

$$(3)\ \sum_{n=0}^{\infty} (n+k)a_n x^{n+k+2} = x^2 \sum_{n=0}^{\infty} (n+k)a_n x^{n+k} = x^2 \sum_{n=k}^{\infty} n a_{n-k} x^n$$

Example 5.1.3. Find a_n satisfying $\sum_{n=1}^{\infty} n a_n x^{n-1} = 2\sum_{n=0}^{\infty} a_n x^n$.

Sol. With shift of index we get

$$\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = 2\sum_{n=0}^{\infty} a_n x^n.$$

So

$$(n+1)a_{n+1} = 2a_n, \quad n = 0, 1, \cdots, .$$

With a_0 arbitrary, we obtain

$$\begin{aligned} a_1 &= 2a_0 \\ a_2 &= a_1 = 2a_0 \\ &\cdots \\ a_{n+1} &= \frac{2}{n+1} \cdot \frac{2}{n} \cdots \frac{2}{1} a_0 = \frac{2^{n+1}}{(n+1)!} a_0. \end{aligned}$$

□

5.1.2 Power Series Solution

Series solution method is useful when the coefficients are not constant or when methods introduced in the previous sections does not work. For example we have a **Bessel equation**

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad (\nu \geq 0)$$

or **Legendre equation**

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (\alpha \geq 0).$$

Given a second order linear differential of the form

$$A(x)y'' + B(x)y' + C(x)y = 0, \tag{5.1}$$

we assume $A(x), B(x), C(x)$ are polynomials without common factors. Divide (5.1) by $A(x)$ we get

$$\boxed{y'' + P(x)y' + Q(x)y = 0}$$

Here $P(x) = B(x)/A(x), Q(x) = C(x)/A(x)$.

Definition 5.1.4. If $P(x)$ and $Q(x)$ are analytic on an interval about x_0, then x_0 is called an (**ordinary point**). Otherwise, x_0 is a **singular point**.

Then by Theorem 2.**??** there exists unique solution satisfying ICs, $y(x_0) = y_0$, $y'(x_0) = y_0'$. If x_0 is a singular point we do not have the result of Theorem 2.**??**. Still some solution can be found.

Polynomial coeff.

primarily, we will consider polynomial coeff.

Theorem 5.1.5. *If x_0 is an ordinary point of the DE. (5.1), then there always exist two linearly independent solutions which can be expressed as a power series $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$, which converge at least on some interval $|x - x_0| < R$.*

Example 5.1.6. Find a series solution

$$y' + y = 0, \quad -\infty < x < \infty.$$

Sol. Let $y = \sum\limits_{n=0}^{\infty} a_n x^n$ and substitute

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Shift index gives

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}((n+1)a_{n+1}+a_n)x^n = 0.$$

Thus we have

$$(n+1)a_{n+1} + a_n = 0, \quad n = 0, 1, 2, \cdots .$$

From this we see

$$\begin{aligned} a_1 &= -a_0 & & \\ a_2 &= -\tfrac{a_1}{2} &&= \tfrac{a_0}{2} \\ a_3 &= -\tfrac{a_2}{3} &&= -\tfrac{a_0}{3\cdot 2} \\ &\cdots && \\ a_n &= -\tfrac{a_{n-1}}{n} &&= \tfrac{(-1)^n a_0}{n!} \end{aligned}$$

and hence

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = a_0 e^{-x}.$$

Example 5.1.7. Find expansion of $\frac{1}{x^2-2x+5}$ at $x = 1$. What is radius of convergence ?

Sol. For $\left|\frac{x-1}{2}\right| < 1$

$$\begin{aligned} \frac{1}{x^2-2x+5} = \frac{1}{(x-1)^2+4} &= \frac{1}{4}\frac{1}{1+\left(\frac{x-1}{2}\right)^2} \\ &= \frac{1}{4}\left(1 - \left(\frac{x-1}{2}\right)^2 + \left(\frac{x-1}{2}\right)^4 + \cdots + (-1)^n\left(\frac{x-1}{2}\right)^{2n} + \cdots\right). \end{aligned}$$

Hence radius of convergence is 2.

Example 5.1.8 (Airy's equation)**.** Solve $y'' + xy = 0$. (or $y'' - xy = 0$)

Solution. Substitute the following into the DE.

$$y(x) = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

$$y'' + xy = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}. \tag{5.2}$$

Shift and adjust the index we get

$$y'' + xy = c_2 + \sum_{k=1}^{\infty}[(k+1)(k+2)c_{k+2} + c_{k-1}]x^k = 0. \tag{5.3}$$

Comparing the coefficients, we see $c_2 = 0$ and

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)},\ k = 1, 2, \cdots . \tag{5.4}$$

This recurrence relation determines c_k:

$$\begin{aligned}
k = 1 : c_3 &= -\frac{c_0}{2 \cdot 3} \\
k = 2 : c_4 &= -\frac{c_1}{3 \cdot 4} \\
k = 3 : c_5 &= -\frac{c_2}{4 \cdot 5} = 0 \\
k = 4 : c_6 &= -\frac{c_3}{5 \cdot 6} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6} \\
k = 5 : c_7 &= -\frac{c_4}{6 \cdot 7} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7} \\
&= \cdots
\end{aligned}$$

$$y = c_0 + c_1 x + 0 - \frac{c_0}{2 \cdot 3}x^3 - \frac{c_1}{3 \cdot 4}x^4 + 0 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6}x^6 \tag{5.5}$$

$$+\frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7}x^7 + 0 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}x^9 + \cdots \tag{5.6}$$

Grouping terms of c_0 and c_1 we have two type of solution:

$$\begin{aligned}
y_1 &= 1 - \frac{1}{2 \cdot 3}x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}x^6 - \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}x^9 + \cdots \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2 \cdot 3 \cdots (3n-1)(3n)} x^{3k} \\
y_2 &= x - \frac{1}{3 \cdot 4}x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}x^7 - \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}x^{10} + \cdots \\
&= x + \sum_{k=1}^{\infty} \frac{(-1)^k}{3 \cdot 4 \cdots (3n)(3n+1)} x^{3k+1}.
\end{aligned}$$

Example 5.1.9. Solve $(x^2 + 1)y'' + xy' - y = 0$.

Solution. This equation has a singularity at $x = \pm i$ and power series will converge for $|x| < 1$ only. With $y(x) = \sum_{n=0}^{\infty} c_n x^n$ we find

$$\begin{aligned}
&(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n \\
&= -c_0 + \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty}[kc_k - c_k]x^k \\
&= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k \\
&= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0.
\end{aligned}$$

Comparing the coefficients, we see $2c_2 - c_0 = 0,\ c_3 = 0$ and

$$(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}, \quad k = 2, 3, \cdots . \tag{5.7}$$

Thus,

$$\begin{aligned}
c_2 &= \frac{1}{2}c_0, \quad c_3 = 0 \\
c_{k+2} &= \frac{1-k}{k+2}c_k, k = 2, 3, \cdots
\end{aligned}$$

Hence

$$\begin{aligned}
c_4 &= -\frac{1}{4}c_2 = -\frac{1}{2\cdot 4}c_0 = -\frac{1}{2^2\cdot 2!}c_0 \\
c_5 &= -\frac{2}{5}c_3 = 0 \\
c_6 &= -\frac{3}{6}c_4 = \frac{3}{2\cdot 2\cdot 4\cdot 6}c_0 = \frac{1\cdot 3}{2^3\cdot 3!}c_0 \\
c_7 &= -\frac{4}{7}c_5 = 0 \\
&= \cdots .
\end{aligned}$$

Note that there is no conditions or relation on c_1(free). So $y = c_1 x$ is a solution. Grouping terms of c_0 and c_1 we have the solution $y = c_0 y_1 + c_1 y_2$:

$$y_1 = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n},\ y_2 = x,\ \text{for } |x| < 1.$$

Example 5.1.10 (Three term recurrence)**.** Consider the DE $y'' - (1+x)y = 0$.

We obtain $c_2 = c_0/2$ and

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, k = 1, 2, 3, \cdots$$

Here c_0 and c_1 are free. To simplify, first choose $c_0 \neq 0$ and $c_1 = 0$. Next choose $c_0 = 0$ and $c_1 \neq 0$.

(1) $c_0 \neq 0$ and $c_1 = 0$

$$\begin{aligned}
c_2 &= \frac{1}{2}c_0 \\
c_3 &= \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_0}{2 \cdot 3} = \frac{1}{6}c_0 \\
c_4 &= \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_0}{2 \cdot 3 \cdot 4} = \frac{1}{24}c_0 \\
c_5 &= \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_0}{4 \cdot 5}(\frac{1}{6} + \frac{1}{2})
\end{aligned}$$

(2) $c_0 = 0$ and $c_1 \neq 0$

$$\begin{aligned}
c_2 &= \frac{1}{2}c_0 = 0 \\
c_3 &= \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_1}{2 \cdot 3} = \frac{1}{6}c_1 \\
c_4 &= \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_1}{2 \cdot 3 \cdot 4} = \frac{1}{12}c_1 \\
c_5 &= \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_1}{4 \cdot 5 \cdot 6} = \frac{1}{120}c_1
\end{aligned}$$

Hence

$$\begin{aligned}
y_1 &= 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots \\
y_2 &= x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots
\end{aligned}$$

Example 5.1.11 (Non polynomial coefficients)**.** $y'' + (\cos x)y = 0$.

$$
\begin{aligned}
0 &= y'' + (\cos x)y \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \sum_{n=0}^{\infty} c_n x^n \\
&= 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)(c_0 + c_1 x + c_2 x^2 + \cdots) \\
&= 2c_2 + c_0 + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3 + \cdots .
\end{aligned}
$$

Comparing the coefficients, we obtain

$$2c_2 + c_0 = 0,\ 6c_3 + c_1 = 0,\ 12c_4 + c_2 - \frac{1}{2}c_0 = 0,\ 20c_5 + c_3 - \frac{1}{2}c_1 = 0, \cdots ,$$

which gives

$$c_2 = -\frac{1}{2}c_0,\ c_3 = -\frac{1}{6}c_1,\ c_4 = \frac{1}{12}c_0,\ c_5 = \frac{1}{30}c_1, \cdots .$$

$$y_1 = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots , \quad y_2 = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots .$$

nonpolynomial coefficients

In this case we can use Taylor expansion to find a series solution.

nonhomogeneous term

There can be nonhomogeneous term in DE. (5.1)

Exercise 5.1.12. (1) For each of the following DEs, find a series solution at $x = 0$.

(a) $y'' - y = 0$

(b) $y'' - 2xy = 0$

(c) $y'' - 2xy' + y = 0$

(d) $(x^2 - 3)y'' - 2xy' + 2y = 0$

(e) $xy'' - (x + 1)y = 0$

(f) $y'' - xy' - (x + 2)y = 0$

(g) $y'' + e^x y' + 2y = 0$

(h) $y'' - 2xy' + 2y = 0$

(i) $(1 - x^2)y'' - xy' + \alpha^2 y = 0$(Chebyshev equation)

5.2 Solution near Singular Points

Definition 5.2.1. Consider a singular DE, i.e., $A(x_0) = 0$

$$A(x)y'' + B(x)y' + C(x)y = 0. \tag{5.8}$$

By dividing we have the following form

$$y'' + P(x)y' + Q(x)y = 0, \tag{5.9}$$

where $P(x)$ or $Q(x)$ is singular at x_0.

We further classify it into these cases: If both $p(x) = (x - x_0)P(x), q(x) = (x-x_0)^2Q(x)$ are analytic at $x = 0$, then x_0 is called a **regular singular point**. In this case, multiplying by $(x - x_0)^2$ we have

$$\begin{aligned} & (x - x_0)^2y'' + (x - x_0)^2P(x)y' + (x - x_0)^2Q(x)y & (5.10)\\ = \quad & (x - x_0)^2y'' + (x - x_0)p(x)y' + q(x)y = 0, & (5.11)\end{aligned}$$

where $p(x)$ and $q(x)$ are analytic.

Example 5.2.2 (Regular Singular point)**.** (1) $x^2y'' + 4xy' + y = 0$. $x = 0$ is a reg. sing. point.

(2) $(x - 3)^2y'' + 7(x - 3)y' + e^xy = 0$. $x = 3$ is a reg. sing. point

(3) $(x - 3)^2y'' + y' + e^xy = 0$. $x = 3$ is NOT a reg. sing. point.

Frobenius Method

Theorem 5.2.3 (Frobenius(1849-1917) Theorem)**.** *If x_0 is a regular singular point, then there exists at least one nonzero solution of the form*

$$y(x) = (x - x_0)^r \sum_{n=0}^{\infty} c_n(x - x_0)^n, \quad a_0 \neq 0, \tag{5.12}$$

where r (not nec. an integer) is a constant to be determined.

Remark 5.2.4. The Frobenius method guarantees only one solution of such form exists. For the other solution, in general, the Theorem does not give any guarantee.

For simplicity, we assume $x_0 = 0$ from now on.

Example 5.2.5. [Distinct roots, $r_1 - r_2$ not integer] Find a series solution of $2xy'' + y' + xy = 0$.

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r} \tag{5.13}$$

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \\ &= [r c_0 x^{r-1} + (r+1) c_1 x^r + \cdots + (n+r) c_n x^{n+r-1} + \cdots] \\ &= x^{r-1} [r c_0 + (r+1) c_1 x + \cdots + (n+r) c_n x^n + \cdots] \end{aligned} \tag{5.14}$$

$$\begin{aligned} y'' &= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} \\ &= [r(r-1) c_0 x^{r-2} + (r+1) r c_1 x^{r-1} + \cdots + (n+r)(n+r-1) c_n x^{n+r-2} + \cdots] \\ &= x^{r-2} [r(r-1) c_0 + (r+1) r c_1 x + \cdots + (n+r)(n+r-1) c_n x^n + \cdots] \end{aligned} \tag{5.15}$$

Subst. into the differential equation $2xy'' + y' + xy = 0$,

$$\begin{aligned} &2x^{r-1}[r(r-1)c_0 + (r+1)rc_1 x + (r+2)(r+1)c_2 x^2 + \cdots + (n+r)(n+r-1)c_n x^n + \cdots] \\ &+ x^{r-1}[rc_0 + (r+1)c_1 x + (r+2)c_2 x^2 + \cdots + (n+r)c_n x^n + \cdots] \\ &+ x^{r+1}[c_0 + c_1 x + \cdots + c_n x^n + \cdots] = 0. \end{aligned}$$

Compare coefficients of x^{r-1}, x^r and x^{r+1}, we get :

$$[2r(r-1) + r]c_0 = 0 \tag{5.16}$$

$$[2(r+1)r + (r+1)]c_1 = 0 \tag{5.17}$$

$$2(n+r)(n+r-1)c_n + (n+r)c_n + c_{n-2} = 0, \ n \geq 2. \tag{5.18}$$

So we obtain $r = 0, \frac{1}{2}$. The equation $F(r) = r(2r-1) = 0$ is called the **indicial equation**. In this case we have

Distinct roots, $r_1 - r_2$ not integer

- Coeff. x^r: $[2(r+1)r + (r+1)]c_1 = (2r+1)(r+1)c_1 = 0 \Rightarrow c_1 = 0$.
- Coeff. x^{n+r-1}: $2(n+r)(n+r-1)c_n + (n+r)c_n + c_{n-2} = 0$. $(n \geq 2)$

Hence

$$c_n = \frac{-c_{n-2}}{(n+r)(2n+2r-1)}, \quad n \geq 2.$$

$c_1 = c_3 = c_5 = \cdots = 0$.

Two solutions are as follows:

(1) $r_1 = \frac{1}{2}$: Use recurrence relation

$$c_n = \frac{-c_{n-2}}{n(2n+1)}$$

$$\begin{aligned}
c_2 &= -\frac{c_0}{2\cdot 5} \\
c_4 &= -\frac{c_2}{4\cdot 9} &&= \frac{c_0}{(2\cdot 5)(4\cdot 9)} \\
c_6 &= -\frac{c_4}{6\cdot 13} &&= -\frac{c_0}{(2\cdot 5)(4\cdot 9)(6\cdot 13)} \\
&\cdots && \cdots \\
c_{2k} &= -\frac{c_{2k-2}}{2k(4k+1)} &&= \frac{(-1)^k c_0}{(2\cdot 4\cdots 2k)(5\cdot 9\cdot 13\cdots(4k+1))} \\
&= \cdots &&= \frac{(-1)^k c_0}{2^k k!(5\cdot 9\cdot 13\cdots(4k+1))}.
\end{aligned}$$

(2) $r = 0$:

$$c_n = \frac{-c_{n-2}}{n(2n-1)}$$

$$\begin{aligned}
c_2 &= -\frac{c_0}{2\cdot(2\cdot 2-1)} &&= -\frac{c_0}{2\cdot 3} \\
c_4 &= -\frac{c_2}{4\cdot(2\cdot 4-1)} &&= \frac{c_0}{(2\cdot 3)(4\cdot 7)} \\
c_6 &= -\frac{c_4}{6\cdot(2\cdot 6-1)} &&= -\frac{c_0}{(2\cdot 3)(4\cdot 7)(6\cdot 11)} \\
&\cdots \\
c_{2k} &= -\frac{c_{2k-2}}{2k(4k-1)} &&= \frac{(-1)^k c_0}{(2\cdot 4\cdots 2k)(3\cdot 7\cdots(4k-1))} \\
& &&= \frac{(-1)^k c_0}{2^k k!(3\cdot 7\cdot 11\cdots(4k-1))}.
\end{aligned}$$

Hence

$$\begin{aligned}
y &= c_0 x^{1/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!(5\cdot 9\cdot 13\cdots(4k+1))}x^{2k} \\
&\quad + c_1\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k k!(3\cdot 7\cdot 11\cdots(4k-1))}x^{2k}.
\end{aligned}$$

Remark 5.2.6. We see

$$\boxed{c_n = -\frac{c_{n-2}}{F(n+r)}, \quad (n \geq 2)}$$

Now we present more general case: Let $p(x) = \sum_{n=0}^{\infty} a_n x^n$, $q(x) = \sum_{n=0}^{\infty} b_n x^n$ and consider

$$x^2 y'' + xp(x)y' + q(x)y = 0. \tag{5.19}$$

The derivatives of y are

$$\begin{aligned}
y' &= \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\
&= [rc_0 x^{r-1} + (r+1)c_1 x^r + \cdots + (n+r)c_n x^{n+r-1} + \cdots] \\
&= x^{r-1}[rc_0 + (r+1)c_1 x + \cdots] \\
y'' &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} \\
&= [r(r-1)c_0 x^{r-2} + (r+1)rc_1 x^{r-1} + \cdots + (n+r)(n+r-1)c_n x^{n+r-2} + \cdots] \\
&= x^{r-2}[r(r-1)c_0 + (r+1)rc_1 x + \cdots].
\end{aligned}$$

Subst. these into (5.10) together with $p(x), q(x)$, and divide by x^r to obtain

$$\begin{aligned}
[r(r-1)c_0 + (r+1)rc_1 x + \cdots] + [a_0 + a_1 x + \cdots][rc_0 + (r+1)c_1 x + \cdots] \\
+[b_0 + b_1 x + \cdots][c_0 + c_1 x + \cdots] = 0.
\end{aligned}$$

Comparing coefficient, we see

$$[r(r-1) + a_0 r + b_0]c_0 = 0.$$

Here c_0 is arb. Hence we obtain the following **indicial equation**.

$$\boxed{F(r) := r(r-1) + a_0 r + b_0 = 0}$$

Denote the zeros by r_1, r_2. Coefficients of x^{r+n} :

$$\boxed{F(r+n)c_n + \sum_{k=0}^{n} c_k[(r+k)a_{n-k} + b_{n-k}] = 0, \quad n \geq 1} \tag{5.20}$$

Example 5.2.7. [multiple roots, or $r_1 - r_2$ is an integer] The solution is complicated.

Theorem 5.2.8. *Assume the coefficients of the DE.*

$$x^2 y'' + xp(x)y' + q(x)y = 0 \tag{5.21}$$

have power series

$$p(x) = \sum_{n=0}^{\infty} a_n x^n, \quad q(x) = \sum_{n=0}^{\infty} b_n x^n$$

convergent for $|x| < \rho$ *and the roots of indical equation are* $r_1, r_2 (r_1 \geq r_2)$. *Then the equation (5.21) has the following type of solution which converges on* $|x| < \rho$.

(1) r_1, r_2 *are distinct and* $r_1 - r_2$ *not integer : There exists always two linearly independent solution of the form*

$$y_1(x) = |x|^{r_1}(1 + c_1(r_1)x + c_2(r_1)x^2 + \cdots)$$

Here $c_n(r_1)$ *is given by (5.20) with* $(c_0 = 1, r = r_1)$.

$$y_2(x) = |x|^{r_2}(1 + c_1(r_2)x + c_2(r_2)x^2 + \cdots)$$

Here $c_n(r_2)$ *is given by (5.20) with* $(c_0 = 1, r = r_2)$.

(2) $r_1 - r_2 = N$ *integer:*

$$\begin{aligned} y_1(x) &= |x|^{r_1}(c_0 + c_1 x + \cdots) \\ y_2(x) &= C y_1(x) \ln x + |x|^{r_2}(b_0 + b_1 x + b_2 x^2 + \cdots). \end{aligned}$$

Here $c_0, c_1, b_0, b_1, \cdots$ *are given by (5.21) and* C *may be zero. If* $C = 0$ *then the two solutions are*

$$y_1(x) = |x|^{r_1}(c_0 + c_1 x + \cdots), \quad y_2(x) = |x|^{r_2}(b_0 + b_1 x + b_2 x^2 + \cdots).$$

(3) $r_1 = r_2$ *: The following type always solution exists.*

$$\begin{aligned} y_1(x) &= |x|^{r}(c_0 + c_1 x + \cdots) \\ y_2(x) &= y_1(x) \ln x + |x|^{r}(b_0 + b_1 x + b_2 x^2 + \cdots) \end{aligned}$$

This is a special case of (2) with $C = 1$ *(the logarithmic term always exists!)*.

How to find the second solution?

With one solution $y_1(x)$ known in the above, you may try $y_2(x) = u(x) y_1(x)$ for the second solution. You will get

$$y_2(x) = y_1(x) \int \frac{e^{-\int^x P(t)dt}}{y_1^2(x)} dx. \tag{5.22}$$

Example 5.2.9. [Double roots]
Solve $xy'' + y' - y = 0$.

Sol. Let $y(x) = x^r \sum_{n=0}^{\infty} c_n x^n$. Then

$$y'(x) = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \text{ and } y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}.$$

Substituting,

$$\begin{aligned} &x^{r-1}[r(r-1)c_0 + (r+1)rc_1 x + \cdots + (n+r)(n+r-1)c_n x^n + \cdots] \\ &+x^{r-1}[rc_0 + (r+1)c_1 x + \cdots + (n+r)c_n x^n + \cdots] \\ &-x^r[c_0 + c_1 x + \cdots + c_n x^n + \cdots] = 0. \end{aligned}$$

Indicial equation is $F(r) = r(r-1) + r = r^2 = 0$. Hence $r = r_1 = r_2 = 0$. Meanwhile the coeff. of x^{n+r-1} satisfies $(n+r)(n+r-1)c_n + (n+r)c_n - c_{n-1} = 0, n \geq 1$. Since $r = 0$ we have

$$c_n = \frac{c_{n-1}}{(n+r)^2} = \frac{c_{n-1}}{n^2}$$

and hence

$$\begin{aligned} c_1 &= c_0 \\ c_2 &= \frac{c_1}{2^2} = \tfrac{c_0}{1^2 2^2} \\ &\cdots \\ c_n &= \frac{c_{n-1}}{n^2} = \tfrac{c_0}{(n!)^2}. \end{aligned}$$

Hence

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

The second solution has the form $y_2(x) = y_1(x) \ln x + x^r \sum_{k=1}^{\infty} A_k x^k$.

Exercise 5.2.10. (1) Find a series solution of the following and when difference is not an integer find the second solution also.

(a) $2x^2y'' - xy' + y = 0$
(b) $xy'' + 3y = 0$
(c) $2x^2y'' + xy' + (x-2)y = 0$
(d) $2xy'' - y' + 2y = 0$
(e) $2x^2y'' - xy' + (x^2+1)y = 0$
(f) $x^2y'' + xy' + (x^2 - \frac{4}{9})y = 0$

(g) $2x^2y'' - x(x-1)y' - y = 0$
(h) $2x^2y'' - x(1+x)y' - y = 0$
(i) $x^2y'' - 2xy' + xy = 0$
(j) $x^2y'' + 5xy' + (3-x^2)y = 0$
(k) $(x+1)^2y'' - (1+x)y' - y = 0$
(l) $xy'' + 3y' + 2x^3y = 0$

5.3 Special Functions

5.3.1 Bessel Functions

The following DE is called the **Bessel's equation of order** ν.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu \geq 0 \tag{5.23}$$

This equation arises in the study of heat equation or wave equation in cylindrical coordinates. Substituting $y = x^r \sum_{n=0}^{\infty} c_n x^n$ into the left of (5.23)

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\
= &\sum_{n=0}^{\infty}\left\{(n+r)(n+r-1)c_n + (n+r)c_n - \nu^2 c_n\right\} x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} \\
= &\left\{r(r-1) + r - \nu^2\right\} c_0 x^r + \left\{(r+1)r + (r+1) - \nu^2\right\} c_1 x^{r+1} \\
&+ \sum_{n=2}^{\infty}\left\{[(n+r)(n+r-1) + (n+r) - \nu^2]c_n + c_{n-2}\right\} x^{n+r}
\end{aligned}$$

Compare coefficients of lowest degree terms,

$$\begin{aligned}
x^r &: (r^2 - \nu^2)c_0 = 0 \\
x^{r+1} &: [(r+1)^2 - \nu^2]c_1 = 0 \\
x^{r+n} &: [(n+r)^2 - \nu^2]c_n + c_{n-2} = 0.
\end{aligned}$$

Indicial equation is $F(r) = r^2 - \nu^2 = 0$, so

$$r = \pm\nu.$$

From the coeff of x^{r+1} (choose $\nu \geq 0$ first)

$$((\pm\nu + 1)^2 - \nu^2)c_1 = (2\nu + 1)c_1 = 0, \tag{5.24}$$

we get $c_1 = 0$. From the coeff of x^{r+n} we get

$$[(n+r)^2 - \nu^2]c_n + c_{n-2} = 0. \tag{5.25}$$

Hence $c_1 = c_3 = c_5 = \cdots = 0$. First consider $r = \nu$. It suffices to consider even terms, so let $n = 2k$. Then from (5.25) we see

$$c_{2k} = -\frac{c_{2k-2}}{2^2 k(k+\nu)}.$$

Hence

$$\begin{aligned}
c_2 &= -\frac{c_0}{2^2 1 \cdot (\nu+1)} \\
c_4 &= -\frac{c_2}{2^2 \cdot 2(\nu+2)} = \frac{c_0}{2^4 (1\cdot 2)(\nu+1)(\nu+2)} \\
c_6 &= -\frac{c_2}{2^2 \cdot 3(\nu+3)} = \frac{-c_0}{2^6 (1\cdot 2\cdot 3)(\nu+1)(\nu+2)(\nu+3)} \\
&\cdots \\
c_{2k} &= \frac{(-1)^k c_0}{2^{2k} k!(\nu+1)(\nu+2)\cdots(\nu+k)}.
\end{aligned}$$

Use a Gamma function defined by

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1.$$

We can easily see the following relation holds :

$$\begin{aligned}
\Gamma(\nu+k+1) &= (\nu+k)\Gamma(\nu+k) \\
&= (\nu+k)(\nu+k-1)\cdots(\nu+1)\Gamma(\nu+1).
\end{aligned}$$

If k is positive integer it holds that

$$\Gamma(k+1) = k!.$$

Since c_0 is arb. we let

$$c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

so that we have

$$c_{2k} = \frac{(-1)^k}{2^{2k+\nu} k! \Gamma(\nu+k+1)}.$$

The solution $y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$ can be written as

$$J_\nu(x) = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+\nu} k! \Gamma(\nu+k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$

When $r = -\nu$, the solution is

$$J_{-\nu}(x) = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-\nu} k! \Gamma(-\nu + k + 1)} \left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(-\nu + k + 1)} \left(\frac{x}{2}\right)^{2k-\nu}.$$

These $J_\nu, J_{-\nu}$ are called **Bessel's function of the first kind** of order ν and $-\nu$.

Remark 5.3.1. (1) If $\nu = 0$ these two functions are the same.

(2) If $\nu > 0$ and the difference $\nu - (-\nu) = 2\nu$ is not a positive integer then by case I above, $J_\nu, J_{-\nu}$ are linearly independent and the gen. solution is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x).$$

(3) The case when ν is a half of odd integer, $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$, $\nu - (-\nu) = 2\nu$ is an odd integer. In this case the two solutions are still linearly independent because the first terms of two solutions are x^ν, $x^{-\nu}$ resp.

(4) The case when ν is an integer, then $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

Example 5.3.2. The gen. solution of the Bessel's equation $x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$ is

$$y(x) = c_1 J_{\frac{1}{2}}(x) + c_2 J_{-\frac{1}{2}}(x).$$

Bessel function of the second kind

If ν is not an integer, the function

$$Y_\nu(x) = \frac{1}{\sin \nu x}[J_\nu(x) \cos \nu\pi - J_{-\nu}(x)] \tag{5.26}$$

is a linearly independent solution of Bessel's equation. Hence the general solution is given by

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

Surprisingly this form of general solution also work when ν is an integer. Define for integer m,

$$Y_m(x) = \lim_{\nu \to m} Y_\nu(x). \tag{5.27}$$

Y_ν is called the **Bessel function of the second kind** of order ν.

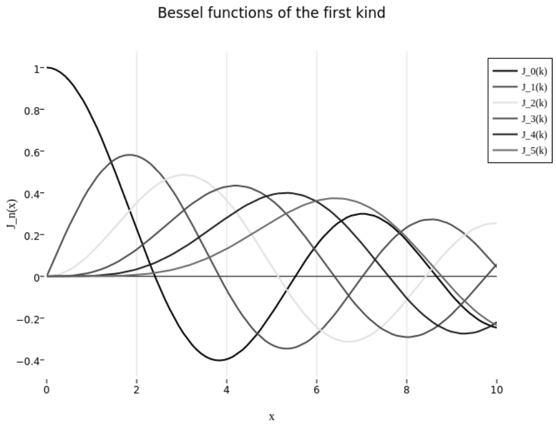

Figure 5.1: Bessel function of the first kind for $J_0, J_1, J_2, \cdots$

A summary for Bessel equation

For any value ν the general solution of the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu \geq 0 \tag{5.28}$$

is given by

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x). \tag{5.29}$$

Example 5.3.3. The gen. solution of the Bessel's equation $x^2y'' + xy' + (x^2 - 16)y = 0$ is

$$y(x) = c_1 J_4(x) + c_2 Y_4(x).$$

DE. that can be solved in terms of Bessel functions

Consider the following DE:

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0, \quad \nu > 0. \tag{5.30}$$

By change of variable $t = \alpha x, \alpha > 0$, we see

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \alpha\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\frac{dy}{dx}\frac{dt}{dx} = \alpha^2\frac{d^2y}{dt^2}.$$

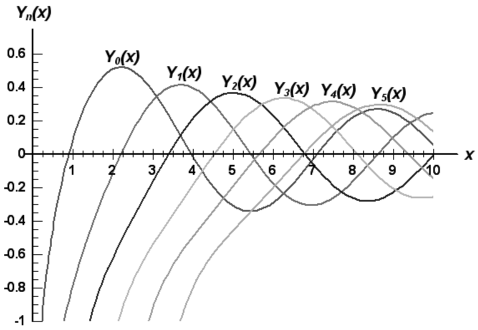

Figure 5.2: Bessel function of the second kind for $n = 0, 1, 2, \cdots$

Thus the Bessel equation becomes

$$\left(\frac{t}{\alpha}\right)^2 \alpha^2 \frac{d^2y}{dt^2} + \left(\frac{t}{\alpha}\right) \alpha \frac{dy}{dt} + (t^2 - \nu^2)y = 0 \Rightarrow t^2y'' + ty' + (t^2 - \nu^2)y = 0, \quad \nu > 0. \tag{5.31}$$

The solution is now known as

$$y(t) = c_1 J_\nu(t) + c_2 Y_\nu(t).$$

Substitution $t = \alpha x$ gives

$$y(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x).$$

This equation is called the **parametric Bessel equation of order** ν.

Modified Bessel equation

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0, \quad \nu > 0. \tag{5.32}$$

This is the same as Bessel equation if we set $t = ix$, $i^2 = -1$. So

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - \nu^2)y = 0, \quad \nu > 0.$$

The solution of this DE $y(t) = c_1 J_\nu(t) + c_2 Y_\nu(t)$. Hence the solution of modified Bessel equation is now complex valued $c_1 J_\nu(ix) + c_2 Y_\nu(ix)$.

The **modified Bessel function of the first kind** is defined as

$$I_\nu(x) = i^{-\nu} J_\nu(ix), \ \nu \neq \text{ integer} \tag{5.33}$$

and the general solution is given by

$$c_1 I_\nu(x) + c_2 I_{-\nu}(x).$$

If ν is an integer these two functions are not linearly independent. Hence we introduce another function

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi x} \tag{5.34}$$

and for integer $\nu = n$, we have

$$K_n(x) = \lim_{nu \to n} K_\nu(x).$$

For any ν, the general solution is now

$$c_1 I_\nu(x) + c_2 K_\nu(x). \tag{5.35}$$

Similarly, parametric form of modified Bessel function of the second kind can be defined.

$$x^2 y'' + xy' - (\alpha^2 x^2 + \nu^2) y = 0, \tag{5.36}$$

whose general solution is given by

$$c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x). \tag{5.37}$$

Bessel Functions of Half-integral Order

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \tag{5.38}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \tag{5.39}$$

Theorem 5.3.4. *We have the following*

(1) For $m = 0, 1, 2, \cdots$, $J_{-m}(x) = (-1)^m J_m(x)$.

(2) $J_m(-x) = (-1)^m J_m(x)$.

(3) $J_m(0) = 0$ if $m > 0$ and $J_0(0) = 1$.

(4) $\lim_{x \to 0^+} Y_m(x) = -\infty$.

5.3.2 Legendre Equation

The following type of DE. is called the **Legendre's equation**.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n \text{ real} \tag{5.40}$$

The solution of this equation is called the **Legendre function**. Let

$$y = \sum_{k=0}^{\infty} c_k x^k \tag{5.41}$$

and substitute into (5.40). With $\alpha = n(n+1)$ we have

$$\begin{aligned}
&(1-x^2)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - 2x\sum_{k=1}^{\infty} kc_k x^{k-1} + \alpha\sum_{k=0}^{\infty} c_k x^k \\
&= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)c_k x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \alpha\sum_{k=0}^{\infty} c_k x^k = 0
\end{aligned}$$

and with shift of index we have

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)c_k x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \alpha\sum_{k=0}^{\infty} c_k x^k = 0.$$

Now

(1) Coeff. of 1 : $2c_2 + n(n+1)c_0 = 0$

(2) Coeff. of x : $6c_3 + [-2 + n(n+1)]c_1 = 0$

(3) Coeff. of x^k :

$$(k+2)(k+1)c_{k+2} + [-k(k-1) - 2k + n(n+1)]c_k = 0.$$

Thus

$$c_{k+2} = -\frac{(n-k)(n+k+1)}{(k+2)(k+1)}c_k, \quad k = 0, 1, \cdots, \tag{5.42}$$

where c_0, c_1 are arbitrary. For $k = 0, 1, 2, \cdots$ we see

$$\begin{aligned}
c_2 &= -\frac{n(n+1)}{2!}c_0 \\
c_3 &= -\frac{(n-1)(n+2)}{3!}c_1 \\
c_4 &= -\frac{(n-2)(n+3)}{4\cdot 3}c_2 &&= \frac{(n-2)n(n+1)(n+3)}{4!}c_0 \\
c_5 &= -\frac{(n-3)(n+4)}{5\cdot 4}c_3 &&= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1 \\
&\cdots &&\cdots
\end{aligned}$$

Set $c_0 = 1$ and collect even number terms

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \cdots \tag{5.43}$$

$a_1 = 1$ and collect odd number terms

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots \tag{5.44}$$

Here y_1, y_2 are independent and interval of convergence is $|x| < 1$. Thus the general solution of (5.40) is given by

$$y(x) = c_1 y_1(x) + c_1 y_2(x).$$

Note that if n is even integer then the series for y_1 terminates (hence becomes a polynomial) and if n is odd integer then the series for y_2 terminates.

Legendre Polynomials

A special case of Legendre function when n is a natural number: If $k = n$ in (5.42) then $c_{n+2} = 0, c_{n+4} = 0, c_{n+6} = 0, \cdots$. If n is even $y_2(x)$ is a polyn. of degree n and if n is odd then $y_1(x)$ is a polyn. of degree n. These are **Legendre polynomials**. Solve (5.42) for c_k then using

$$c_k = -\frac{(k+2)(k+1)}{(n-k)(n+k+1)}c_{k+2}, \quad k \le n-2. \tag{5.45}$$

all terms are expressed in terms of c_n. Since c_0 is arbitrary, we may set

$$c_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}, \quad n = 1, 2, 3, \cdots \tag{5.46}$$

so that

$$\begin{aligned} c_{n-2} &= -\frac{n(n-1)}{2(2n-1)}c_n = -\frac{n(n-1)(2n)!}{2(2n-1)2^n(n!)^2} \\ &= -\frac{n(n-1)2n(2n-1)(2n-2)!}{2(2n-1)2^n n(n-1)! n(n-1)(n-2)!} \\ &= -\frac{(2n-2)!}{2^n(n-1)!(n-2)!}. \end{aligned}$$

Similarly, we see

$$\begin{aligned} c_{n-4} &= -\frac{(n-2)(n-3)}{4(2n-3)}c_{n-2} = \frac{(2n-4)!}{2^n 2!(n-2)!(n-4)!} \\ &\cdots \\ c_{n-2k} &= (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}. \end{aligned}$$

Let $M = [\frac{n}{2}]$

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{M} (-1)^k \frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \\ &= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1!(n-1)!(n-2)!} x^{n-2} + \cdots \end{aligned}$$

For $n = 0, 1, 2, \cdots, 5$ we see P_n are:

$P_0(x) = 1,$	$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1),$	$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$	$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

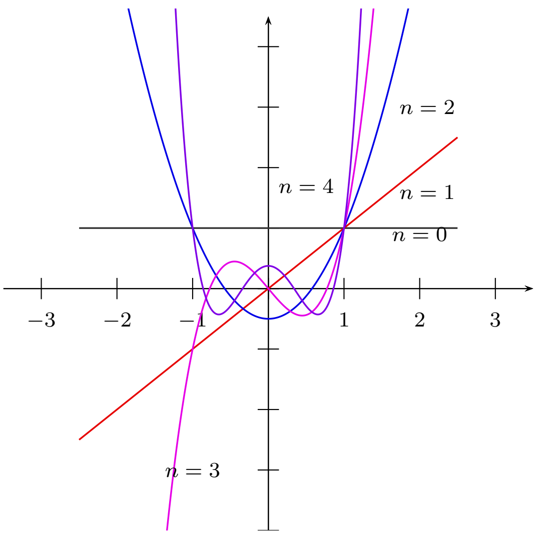

Figure 5.3: Legendre polynomial $n = 0, 1, \cdots, 4$

Properties of Legendre polynomial

(1) $P_n(-x) = (-1)^n P_n(x)$

(2) $P_n(1) = 1$

(3) $P_n(-1) = (-1)^n$

(4) $P_n(0) = 0,\ n \quad \text{odd}$

(5) $P_n'(0) = 0,\ n$ even

Recurrence relation

We state without proof that

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0 \qquad (5.47)$$

for $k = 1, 2, \cdots$.

Another useful formula is by Rodrigues:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \;\; n = 0, 1, 2, \cdots . \qquad (5.48)$$